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ON CLOSEDNESS OF THE RANGE OF THE OPERATOR $X \mapsto TX - XT$ DEFINED ON $\mathcal{C}_2(\mathcal{H})$ WHEN T IS *M*-HYPONORMAL

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Abstract

We show how a proof of Stampfli can be extended to prove that the operator $X \mapsto TX - XT$ defined on the Hilbert-Schmidt class, when T is an M-hyponormal operator, has a closed range, if and only if $\sigma(T)$ is finite.

1. Introduction

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, let $\mathcal{L}(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$. The Hilbert-Schmidt class, denoted by $\mathcal{C}_2(\mathcal{H})$, is a Hilbert space with the $\|\cdot\|_2$-norm that arises from the inner product $\langle X, Y\rangle = \operatorname{tr}(XY^*)$, where tr is the scalar-valued trace. For $T \in \mathcal{L}(\mathcal{H})$, define $\Delta_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by $\Delta_T(X) = TX - XT$, and let $\sigma(T)$ denote the spectrum of T. Let the range of a linear operator S be denoted by $\mathcal{R}(S)$. For a normal operator $N \in \mathcal{L}(\mathcal{H})$, Anderson and Foiaş [1] proved that $\mathcal{R}(\Delta_N)$ is norm closed, if and

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only if $\sigma(N)$ is a finite set. In [3], Stampfli extended this result to the class of hyponormal operators.

Theorem A ([3])**.** *Let* $T \in \mathcal{L}(\mathcal{H})$ *be a hyponormal operator. Then* $\mathcal{R}(\Delta_T)$ *is norm closed, if and only if* $\sigma(T)$ *is finite.*

In fact, Stampfli provided a proof of the "only if " implication which can be extended to a larger class of operators than hyponormal operators. For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\sigma_{nap}(T)$ denote its *normal approximate point spectrum*, that is, the set of those $\lambda \in \mathbb{C}$ for which there exists an orthonormal sequence $\{\phi_n\}_n$ in $\mathcal{H}$ such that

$$\|(T-\lambda)\phi_n\| + \|(T-\lambda)^*\phi_n\| \to 0.$$

Define the class $\mathcal{G}(\mathcal{H})$ as follows:

$$\mathcal{G}(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) \mid \sigma_{nap}(T) \text{ is an infinite set}\}.$$

Some classes of hyponormal related operators, such as M-hyponormal operators, i.e.,

$$m \cdot \|(T-\lambda)^*\phi\| \le \|(T-\lambda)\phi\|, \ \ \forall\, \phi \in \mathcal{H}, \text{ and } \forall\, \lambda \in \mathbb{C}, \text{ for some } m > 0,$$

have spectrum that is finite or they belong to $\mathcal{G}(\mathcal{H})$. In particular, the hyponormal operators (that is, 1-hyponormal) have this property.

In [2], Stampfli proved the following lemma which will be used in Section 2.

Lemma B. *Let* $T \in \mathcal{G}(\mathcal{H})$ *and let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence of distinct points of* $\sigma_{nap}(T)$*. Then for any sequence* $\{\varepsilon_n\}_{n=1}^{\infty}$ *of positive numbers converging to zero, there exists an orthonormal sequence* $\{\phi_n\}_{n=1}^{\infty}$ *of vectors in* $\mathcal{H}$ *such that*

$$\|(T-\lambda_n)\phi_n\| + \|(T-\lambda_n)^*\phi_n\| < \varepsilon_n, \quad \text{for } n = 1, 2, \ldots, \text{ and} \tag{1}$$

$$\langle \phi_n, T\phi_k \rangle = 0, \text{ for } k = 1, \ldots, n-1. \tag{2}$$

2. The Closedness of the Range of $\Delta_T^{(2)}$

The operator Δ_T defined on the Hilbert-Schmidt class will be denoted in the remainder of this note by $\Delta_T^{(2)}$, that is, $\Delta_T^{(2)} : \mathcal{C}_2(\mathcal{H}) \to \mathcal{C}_2(\mathcal{H})$, $\Delta_T^{(2)}(X) = TX - XT$. Let $H^M(\mathcal{H})$ denote the set of M-hyponormal operators.

Proposition 1. *Let $T \in H^M(\mathcal{H})$. If $\sigma(T)$ is finite, then $\mathcal{R}(\Delta_T^{(2)})$ is closed.*

Proof. It is well known that an operator $T \in H^M(\mathcal{H})$ with finite spectrum is normal. Indeed, for the such an operator, the restriction to an invariant subspace $\mathcal{M}$ belongs to $H^M(\mathcal{M})$. On the other hand, if $T \in H^M(\mathcal{H})$ with $\sigma(T) = \{\lambda\}$, then $T = \lambda I$, (cf. [4]). Thus, we can write $T = \sum_{i=1}^{n_0} \lambda_i E_i$, where E_i's are the spectral projections.

Let X_n and C be in $\mathcal{C}_2(\mathcal{H})$ such that $\|\Delta_T^{(2)}(X_n) - C\|_2 \to 0$. Therefore, $\Delta_T(X_n) - C \to 0$ in the $\mathcal{L}(\mathcal{H})$ norm, and according to Theorem A, there exists $X^0 \in \mathcal{L}(\mathcal{H})$ such that $C = TX^0 - X^0T$. For an arbitrary $X \in \mathcal{L}(\mathcal{H})$, let $[X_{ij}]$ be the block-matrix representation of X relative to the decomposition $\mathcal{H} = \sum_{i=1}^{n_0} \oplus E_i\mathcal{H}$. Thus

$$C_{ij} = (\lambda_i - \lambda_j)X_{ij}^0,$$

for all $i, j = 1, \dots, n_0$. This implies that each $X_{ij}^0 = \dfrac{1}{(\lambda_i - \lambda_j)} C_{ij}$ is a Hilbert-Schmidt operator. Moreover, X_{ii}^0 can be chosen 0, and thus $X^0 \in \mathcal{C}_2(\mathcal{H})$.

Proposition 2. *Let $T \in \mathcal{G}(\mathcal{H})$. Then $\mathcal{R}(\Delta_T^{(2)})$ is not closed.*

Proof. We will use same notation and circle of ideas as in [2]. Let $\{\lambda_n\}_{n \geq 1}$ be sequence of distinct points of $\sigma_{nap}(T)$ so that $\lambda_n \to \lambda_0$. Let

$$\eta_n = \max\{|\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} \mid j = 1, \dots, n\},$$

and choose a non-increasing sequence $\{\varepsilon_n\}_{n\geq 1}$ so that $0 < \varepsilon_n \leq |\lambda_{n+1} - \lambda_n|^2$, $n \geq 1$, and $\sum_{n\geq 1} \varepsilon_n^2 \eta_n^2 < \infty$. According to Lemma B, there exists an orthonormal sequence $\{\phi_n\}_{n\geq 1}$ that satisfies (1), (2). Let $\mathcal{H}_1 = \vee\{\phi_n \mid n \geq 1\}$, $\mathcal{H}_2 = \mathcal{H}_{-1}^{\perp}$, and let δ_n such that

$$T\phi_n = \mu_n \phi_n + \delta_n \text{ and } \delta_n \perp \phi_n,\ n \geq 1. \tag{3}$$

It results that

$$|\mu_n - \lambda_n| < \varepsilon_n \text{ and } \|\delta_n\| < 2\varepsilon_n,\ n \geq 1. \tag{4}$$

Define $V : \mathcal{H} \to \mathcal{H}$ by $V\phi_n = |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} \phi_{n+1}$, $n \geq 1$, and $Vg = 0$, $g \in \mathcal{H}_2$. Let $\mathcal{M}_n = \vee\{\phi_j \mid j = 1, \dots, n\}$ and let P_n be the orthogonal projection onto $\mathcal{M}_n$ and define $V_n = VP_n$. A tedious calculation shows that

$$\Delta_T(V_n)\phi_j = \begin{cases} v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} - V_n\delta_j, & j \leq n, \\ \\ -V_n\delta_j, & j > n, \end{cases}$$

where $v_j = |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}}$. Denoting $\Delta_T(V_n) - \Delta_T(V_m)$ by $\Delta_T^{n,m}$, then for $n < m$,

$$\Delta_T^{n,m}\phi_j = \begin{cases} 0, & j \leq n, \\ \\ -v_j(\mu_{j+1} - \mu_j)\phi_{j+1} & \\ \\ +v_j\delta_{j+1} + (V_m - V_n)\delta_j, & n < j \leq m, \\ \\ (V_m - V_n)\delta_j, & j > m. \end{cases} \tag{5}$$

Furthermore, from (3), it results that

$$\delta_j \perp \phi_j, \phi_{j+1}, \phi_{j+2}, \dots \tag{6}$$

and from (4),

$$\|V_n\delta_j\| \le 2\eta_j\varepsilon_j, \quad \text{for all } j,\, n \ge 1. \tag{7}$$

We will show next that $\|\Delta_T^{n,m}\|_2 \to 0$, when $m, n \to \infty$. Thus, there exists $C \in \mathcal{C}_2(\mathcal{H})$ such that $\|\Delta_T(V_n) - C\|_2 \to 0$, that is, $C \in \overline{\mathcal{R}(\Delta_T^{(2)})}$.

First, we will show that $\|\Delta_T^{n,m}|_{\mathcal{H}_1}\|_2^2 \to 0$, when $m, n \to \infty$. Indeed,

$$\|\Delta_T^{n,m}|_{\mathcal{H}_1}\|_2^2 = \sum_{j=1}^{\infty}\|\Delta_T^{n,m}\phi_j\|^2 \overset{(5)}{=}$$

$$= \sum_{j=n+1}^{m}\| - v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1} + (V_m - V_n)\delta_j\|^2$$

$$+ \sum_{j=m+1}^{\infty}\|(V_m - V_n)\delta_j\|^2.$$

The first sum of the right hand side of the above can be majorized by

$$2 \cdot \sum_{j=n+1}^{m}\| - v_j(\mu_{j+1} - \mu_j)\phi_{j+1} + v_j\delta_{j+1}\|^2 + 2 \cdot \sum_{j=n+1}^{m}\|(V_m - V_n)\delta_j\|^2.$$

Since $\phi_{j+1} \perp \delta_{j+1}$, we have

$$\|\Delta_T^{n,m}|_{\mathcal{H}_1}\|_2^2 \le 2\Big[\sum_{j=n+1}^{m}(v_j^2|\mu_{j+1} - \mu_j|^2 + v_j^2\|\delta_{j+1}\|^2) + \sum_{j=n+1}^{\infty}\|(V_m - V_n)\delta_j\|^2\Big].$$

According to (7),

$$\|(V_m - V_n)\delta_j\|^2 \le 16\eta_j^2\varepsilon_j^2,$$

and according to (4),

$$v_j^2\|\delta_{j+1}\|^2 \le 4\eta_j^2\varepsilon_{j+1}^2 \le 4\eta_j^2\varepsilon_j^2,$$

and

$$|\mu_{j+1} - \mu_j|^2 \le (2\varepsilon_j + |\lambda_{j+1} - \lambda_j|)^2 \le 8\varepsilon_j^2 + 2|\lambda_{j+1} - \lambda_j|^2,$$

which implies

$$v_j^2|\mu_{j+1}-\mu_j|^2 \leq 8\eta_j^2\varepsilon_j^2 + 2|\lambda_{j+1}-\lambda_j|.$$

Therefore

$$\|\Delta_T^{n,m}|_{\mathcal{H}_1}\|_2^2 \leq c_1 \cdot \sum_{j=n+1}^{\infty}\eta_j^2\varepsilon_j^2 + c_2 \cdot \sum_{j=n+1}^{m}|\lambda_{j+1}-\lambda_j|,$$

where c_1 and c_2 are some constants. After a careful review of the proof, one can see that the sequence $\{\lambda_n\}$ can be assumed to converge fast enough (otherwise, choose a subsequence of it), more precisely

$$\sum\nolimits_{j=n+1}^{m}|\lambda_{j+1}-\lambda_j| \to 0, \text{ when } n, m \to \infty.$$

We show next that $\|\Delta_T^{n,m}|_{\mathcal{H}_2}\|_2^2 \to 0,$ when $m, n \to \infty$. Indeed, we can write

$$T^*\phi_n = \overline{\mu}_n\phi_n + \gamma_n \text{ with } \langle\gamma_n, \phi_n\rangle = 0, \text{ and } \|\gamma_n\| \leq 2\varepsilon_n, \ n \geq 1. \tag{8}$$

Obviously, we can write $T^*\phi_n = \theta_n\phi_n + \gamma_n$ with $\langle\gamma_n, \phi_n\rangle = 0,$ which implies

$$\theta_n = \langle\theta_n\phi_n + \gamma_n, \phi_n\rangle = \langle T^*\phi_n, \phi_n\rangle = \langle\phi_n, T\phi_n\rangle = \langle\phi_n, \mu_n\phi_n + \delta_n\rangle = \overline{\mu}_n,$$

and $\|\gamma_n\| = \|(T^* - \overline{\mu}_n)\phi_n\| \leq \|(T-\lambda_n)^*\phi_n\| + |\overline{\lambda}_n - \overline{\mu}_n| \overset{(1),(4)}{\leq} 2\varepsilon_n.$

For an orthonormal basis $\{\psi_i\}_{i\geq 1}$ of $\mathcal{H}_2$, we will show that

$$\sum_{i=1}^{\infty}\|\Delta_T^{n,m}\psi_i\|^2 \to 0, \text{ when } n, m \to \infty.$$

For each i, write $T\psi_i = \sum_{k=1}^{\infty}a_k^{(i)}\phi_k + w_i$ with $w_i \in \mathcal{H}_2$. Thus

$$V_mT\psi_i = \sum_{k=1}^{m}a_k^{(i)}V_m\phi_k + V_mw_i = \sum_{k=1}^{m}a_k^{(i)}v_k\phi_{k+1}.$$

Since $V_m\psi_i = 0$, we have $\Delta_T(V_m)\psi_i = -V_mT\psi_i$, and consequently, for $n < m$,

$$\Delta_T^{n,m}\psi_i = \sum_{k=n+1}^{m} a_k^{(i)} \upsilon_k \phi_{k+1}.$$

Since the sequence $\{\phi_k\}$ is orthonormal, we have

$$\|\Delta_T^{n,m}\psi_i\|^2 = \sum_{k=n+1}^{m} |a_k^{(i)}|^2 \cdot \upsilon_k^2. \tag{9}$$

Therefore

$$\sum_{i=1}^{\infty}\|\Delta_T^{n,m}\psi_i\|^2 = \sum_{i=1}^{\infty}\sum_{k=n+1}^{m}|a_k^{(i)}|^2 \cdot \upsilon_k^2 = \sum_{k=n+1}^{m}\upsilon_k^2\Big(\sum_{i=1}^{\infty}|a_k^{(i)}|^2\Big).$$

For a fixed k,

$$\sum_{i=1}^{\infty}|a_k^{(i)}|^2 = \sum_{i=1}^{\infty}|\langle T\psi_i,\, \phi_k\rangle|^2 = \sum_{i=1}^{\infty}|\langle \psi_i,\, T^*\phi_k\rangle|^2 \overset{(8)}{=},$$

$$\sum_{i=1}^{\infty}|\langle \psi_i,\, \bar{\mu}_k\phi_k + \gamma_k\rangle|^2 = \sum_{i=1}^{\infty}|\langle \psi_i,\, \gamma_k\rangle|^2 \le \|\gamma_k\|^2 \overset{(8)}{\le} 4\varepsilon_k^2.$$

Consequently, $\sum_{i=1}^{\infty}\|\Delta_T^{n,m}\psi_i\|^2 \le 4\sum_{k=n+1}^{m}\upsilon_k^2 \cdot \varepsilon_k^2 \to 0$, for $n, m \to \infty$.

The operator C is not in $\mathcal{R}(\Delta_T^{(2)})$ since, according to the proof of Theorem A in [3], $C \notin \mathcal{R}(\Delta_T)$.

Theorem 3. *Let $T \in H^M(\mathcal{H})$. Then $\mathcal{R}(\Delta_T^{(2)})$ is closed, if and only if $\sigma(T)$ is finite.*

Proof. If $T \in H^M(\mathcal{H})$ and $\sigma(T)$ is finite, then according to Proposition 1, $\mathcal{R}(\Delta_T^{(2)})$ is closed. Conversely, if $T \in H^M(\mathcal{H})$ has an infinite spectrum, then there are infinitely many distinct points $\{\lambda_n\}_n$ that are either isolated points of the spectrum, in which case they are eigenvalues, or accumulation points of the spectrum, in which case they are in the $\sigma_{ap}(T)$. Since $T \in H^M(\mathcal{H})$, we have $\sigma_p(T), \sigma_{ap}(T) \subseteq \alpha_{nap}(T)$. Thus $T \in \mathcal{G}(\mathcal{H})$, and according to Proposition 2, $\mathcal{R}(\Delta_T^{(2)})$ is not closed.

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